Large deviation properties of on-off intermittency

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The large deviation property of on-off intermittency is investigated by introducing a two-dimensional piecewise linear map, which can be mapped to an *infinite* Markov chain. It is shown that nonanalyticity, in the q-weighted average of the portion of time spent in burst state, appears as a *second-order* phase transition for an interval of control parameter with the bifurcation point of on-off intermittency as its end point.
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I. INTRODUCTION

Intermittency is a highly non-Gaussian temporal behavior commonly observed in various fields of nonlinear sciences. In low-dimensional dynamical systems, intermittent chaos of type I, II, and III was established by Pomeau and Manneville (PM intermittency) [1]. Also in low-dimensional dynamical systems, another type of intermittency was first found by Fujisaka and Yamada in coupled chaotic oscillators [2], and is recently called *on-off intermittency* [3]. This intermittency has been attracting considerable attention. In contrast to PM intermittency, which is associated with instability of periodic orbits, on-off intermittency is observed when the chaotic attractor, which is confined in an invariant sub-manifold of lower dimension than that of the full phase space, loses stability in the transverse direction [4]. Although many studies have been carried out on the universal characteristics of on-off intermittency [5–8], many features, such as its geometric structures, are still to be understood.

The thermodynamic formalism based on the large deviation theory [9] is useful to characterize ergodic properties of dynamical systems [10–12]. However, large deviation properties of on-off intermittency [13] are also not fully understood. In this paper, we shall discuss the large deviation property of the portion of time spent in burst state and show that there appears a *second-order* phase transition. In Sec. II, a model of on-off intermittency, which can be mapped to an *infinite* Markov chain, is introduced and its large deviation property is discussed in Sec. III. A summary and remarks are given in the last section. The Appendix contains detailed calculations.

II. MODIFIED RANDOM WALK MODEL

Let us consider an N-dimensional discrete dynamical system

$$\begin{cases} X_{n+1}=F(X_n,Y_n), \\ Y_{n+1}=G(X_n,Y_n), \end{cases} \tag{1}$$

where $X\in\mathbf{R}^{N_\parallel}$, $Y\in\mathbf{R}^{N_\perp}$, and $N_\parallel+N_\perp=N$. Assume that $G(X,Y)$ is antisymmetric with respect to Y, i.e.,

$$G(X,-Y)=-G(X,Y), \tag{2}$$

so that, $G(X,0)=0$ and the system (1) has the $N_\parallel$-dimensional invariant subspace $S\equiv\{(X,Y)|Y=0\}$. Moreover let us assume that the system

$$X_{n+1}=F(X_n,0), \tag{3}$$

which is the motion restricted within S, has a chaotic attractor $\mathcal{A}$ with an ergodic natural invariant measure.

The linearized motion transverse to S along the dynamics (3) on the chaotic attractor $\mathcal{A}$ is given by

$$\delta Y_{n+1}=D_Y G(X_n,0)\,\delta Y_n, \tag{4}$$

where δY_n denotes the deviation transverse to S and $D_Y G(X,Y)$ denotes the tangent map of $G(X,Y)$ with respect to Y at (X,Y). The transverse stability of this chaotic attractor $\mathcal{A}$ is determined by *the transverse Lyapunov exponent* defined as

$$\lambda_\perp\equiv\lim_{n\to\infty}\frac{1}{n}\log\frac{|\delta Y_n|}{|\delta Y_0|}, \tag{5}$$

which converges to a unique value for almost all X_0 on $\mathcal{A}$ and $\delta Y_0\neq 0$ by the ergodic property of the chaotic attractor $\mathcal{A}$.

If $\lambda_\perp<0$, $\mathcal{A}$ is transversely stable, i.e., infinitesimally small disturbance transverse to S decay in time. If $\lambda_\perp>0$, $\mathcal{A}$ is not transversely stable and does not have global stability, thus a qualitatively different motion appears. This transition of motion at $\lambda_\perp=0$ is called *blow-out bifurcation* [14].

Chaotic behavior in $D_Y G(X_n,0)$ leads to a nonuniform growth of δY_n, i.e., the transverse disturbance may show an intermittent growth and decay in time. If $\lambda_\perp>0$ and there is a global mechanism of reinjection toward $\mathcal{A}$, on-off intermittency is observed.

As a tractable model showing on-off intermittency, we introduce a two-dimensional piecewise linear map on $[0,1]\times[-1,1]$ with $N_\parallel = N_\perp = 1$ by

$$x_{n+1}=\begin{cases} a^{-1}x_n & \text{if} \quad 0\le x_n<a,\\ a'^{-1}(1-x_n) & \text{if} \quad a\le x_n\le 1,\end{cases}$$
$$y_{n+1}=\begin{cases} b^{-1}y_n & \text{if} \quad 0\le |y_n|<b, \quad 0\le x_n<a,\\ by_n & \text{if} \quad 0\le |y_n|<b, \quad a\le x_n\le 1,\\ \mathrm{sgn}(y_n)b'^{-1}(1-|y_n|) & \text{if} \quad b\le |y_n|\le 1,\end{cases} \tag{6}$$

where $0<a,b<1$, $a'=1-a, b'=1-b$, and $\mathrm{sgn}(x)$ denotes the sign of x. Note that y_n does not change its sign along the orbit and thus, in the following, the phase space is restricted to $[0,1]\times[0,1]$. The invariant submanifold S is a unit interval on $y=0$ and the chaotic attractor $\mathcal{A}$ has the uniform natural invariant density on S.

The transverse Lyapunov exponent $\lambda_\perp$ of $\mathcal{A}$ is

$$\lambda_\perp = a\log b^{-1} + a'\log b = (2a-1)\log b^{-1}. \tag{7}$$

Thus the bifurcation point is at $a=a_0\equiv 1/2$ and $\mathcal{A}$ is unstable to the transverse disturbance for $a>a_0$, and not an attractor any more. A time series $\{y_n\}$ for $a=0.505$ and $b=1/e$ is shown in Fig. 1. As can be seen in Fig. 1, the model (6) exhibits characteristics of on-off intermittency.

Let us define rectangles $R_j^i\subset[0,1]\times[0,1]$ as

$$\begin{cases} R_j^0\equiv[0,a)\times(b^{j+1},b^j],\\ R_j^1\equiv[a,1]\times(b^{j+1},b^j],\end{cases} \quad j=0,1,2,\ldots, \tag{8}$$

where $\cup_{i=0}^{1}\cup_{j=0}^{\infty}R_j^i=[0,1]\times[0,1]$ and $R_j^i\cap R_{j'}^{i'}=\emptyset$ if $(i,j)\neq(i',j')$. Since

$$T(R_j^i)=\begin{cases} R_{j+2i-1}^0\cup R_{j+2i-1}^1 & \text{if } j\neq 0,\\ \cup_{i=0}^{1}\cup_{k=0}^{\infty}R_k^i & \text{if } j=0,\end{cases} \tag{9}$$

where T denotes the map (6), the partition $\mathcal{R}=\{R_j^i\}$ gives a *Markov partition*.

Now consider piecewise constant functions such that

$$f(x,y)=\sum_{i=0}^{1}\sum_{j=0}^{\infty}c_j^iR_j^i(x,y), \tag{10}$$

where

$$R_j^i(x,y)\equiv\begin{cases}1 & \text{if } (x,y)\in R_j^i,\\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

Operation of the Frobenius-Perron operator $\mathcal{H}$ on f leads to

$$\mathcal{H}f(x,y)=(c_0^0ab'+c_0^1a'b')\sum_{j=0}^{\infty}R_j(x,y) + \sum_{j=1}^{\infty}[c_j^0abR_{j-1}(x,y)+c_j^1a'b^{-1}R_{j+1}(x,y)], \tag{12}$$

where $R_j(x,y)\equiv R_j^0(x,y)+R_j^1(x,y)$. Moreover, if $f(x,y)$ does not depend on x, i.e., $c_j^0=c_j^1\equiv c_j$, then

$$\mathcal{H}f(y)=c_0b'\sum_{j=0}^{\infty}R_j(x,y) + \sum_{j=1}^{\infty}c_j[abR_{j-1}(x,y)+a'b^{-1}R_{j+1}(x,y)]. \tag{13}$$

Let us denote the masses of the rectangles by

$$p_j\equiv(R_jf)=c_j(R_j)=c_jb'b^j \tag{14}$$

and

$$p_j'\equiv(R_j\mathcal{H}f), \tag{15}$$

where (G) denotes $\int_0^1dx\int_0^1dy\,G(x,y)$, then p_j' is explicitly given as

$$p_j'=\begin{cases} ap_{j+1}+a'p_{j-1}+b'b^jp_0 & \text{if} \quad j\ge 2,\\ ap_{j+1}+b'b^jp_0 & \text{if} \quad j=0,1,\end{cases} \tag{16}$$

which is equivalent to an *infinite* Markov chain and thus we call the system (6) the modified random walk model. If $a>a_0=1/2$, Eq. (16) has the stationary solution

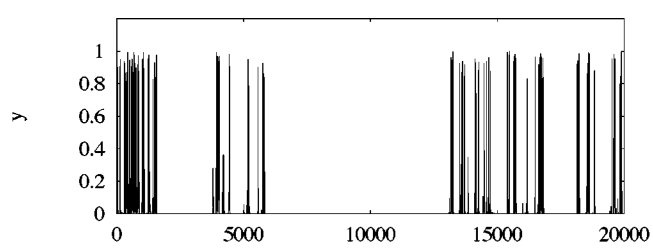

FIG. 1. Time series of y exhibiting on-off intermittency.

$$p_j^s=\begin{cases}(a-a')b'/(1-2a'b') & \text{for } j=0,\\ p_0^s b(a'-ab)^{-1}\{(a'/a)^j-b^j\} & \text{for } j\geq 1,\end{cases} \tag{17}$$

where p_j^s satisfies the normalization condition $\Sigma_{j=0}^{\infty}p_j^s=1$ and the limit $(a'/a)\to b$ is taken in the case $a'=ab$. By using p_j^s, the natural invariant density $\rho(x,y)$ of Eq. (6) is given as

$$\rho(x,y)=\sum_{j=0}^{\infty} p_j^s(R_j)^{-1}R_j(x,y). \tag{18}$$

III. SECOND-ORDER Q-PHASE TRANSITION

In order to investigate the intermittent property of on-off intermittency, let us introduce the indicator of burst and laminar by

$$u(X)\equiv R_0(X), \tag{19}$$

where X denotes (x,y) and $u(X)=0$ and 1 for the laminar and burst states, respectively.

The portion of time spent in the burst state in a time interval of length n is given by

$$u_n(X)\equiv(1/n)\sum_{k=0}^{n-1} u(T^n(X)), \tag{20}$$

where T denotes the map (6). Moreover let us introduce the thermodynamic structure functions [13,15] associated with $u(X)$ by

$$\phi(q)\equiv\lim_{n\to\infty}(1/n)\log\langle e^{nqu_n(X)}\rangle, \tag{21}$$

$$u(q)\equiv d\phi(q)/dq=\lim_{n\to\infty}\langle u_n(X)e^{nqu_n(X)}\rangle/\langle e^{nqu_n(X)}\rangle, \tag{22}$$

and

$$\chi(q)\equiv d^2\phi(q)/dq^2 = \lim_{n\to\infty} n\langle\{u_n(X)-u(q)\}^2e^{nqu_n(X)}\rangle/\langle e^{nqu_n(X)}\rangle, \tag{23}$$

where $-\infty<q<\infty$ and $\langle G(X)\rangle$ denotes the average of $G(X)$ with natural invariant density $\rho(X)$, i.e.,

$$\langle G(X)\rangle=\int dX\rho(X)G(X)=(\rho G). \tag{24}$$

Note that $u(q)$ is a q-weighted average of $u_n(X)$ and enables us to single out some characteristics of invariant sets contained in the chaotic attractor by changing the value of q [15].

After a calculation shown in the Appendix, we obtain the following results. In the case of the critical regime, i.e., $a_0<a<a_c\equiv(1+b^2)^{-1}$,

$$\phi(q)=\begin{cases}\log(2\sqrt{aa'}) & \text{for } -\infty<q<q_c,\\ \log(\lambda(q)) & \text{for } q_c\leq q<\infty,\end{cases} \tag{25}$$

where $q_c=\log\{2(\sqrt{aa'}-ab)/b'\}$ and $\lambda(q)$ is an analytic function, which is defined in the Appendix with $\lambda(q_c)=2\sqrt{aa'}$, $d\lambda(q_c)/dq=0$, and $d^2\lambda(q_c)/dq^2=2\sqrt{aa'}b^{-2}(b-\sqrt{a'/a})^2$. Thus, as shown in Fig. 2, the derivative $\chi(q)$ of the q-weighted average $u(q)$ has a discontinuity at $q=q_c$, exhibiting a *second-order* q-phase transition. The fluctuation spectrum $S(u)$ is shown also in Fig. 2, where $S(u)$ is related to $\phi(q)$ with the Legendre transformation

$$\phi(q)=\max_u\{qu-S(u)\} \quad \text{or} \quad S(u)=\max_q\{qu-\phi(q)\}. \tag{26}$$

The transition point $q_c\to-\infty$ as $a\to a_c-0$ and the nonanalyticity disappears, on the other hand, $q_c\to 0-0$ as $a\to a_0+0$.

Since $u(q=0)=\langle u\rangle=p_0^s=(a-a')b'/(1-2a'b')\propto\epsilon$, $S(u=0)=-\phi(q=-\infty)=-\log(2\sqrt{aa'})\propto\epsilon^2$, for small $\epsilon\equiv a-a_0>0$, and $S(u)$ is parabolic around $u=\langle u\rangle$, a scaling $S(u)\simeq\epsilon^2\sigma(u/\epsilon)$ for $0\leq u\leq\langle u\rangle$ and $u>\langle u\rangle$ with small $u/\langle u\rangle$. Similar scaling has been found to hold in the multiplicative stochastic model of on-off intermittency [13].

As shown in the Appendix, the transition at q_c is brought about by the disappearance of the discrete eigenvalue of the generalized Frobenius-Perron operator associated with $u_n(X)$. The eigenvector for the discrete eigenvalue is localized and may be contributed by localized invariant sets, where the orbit elements visit $R_0=R_0^0\cup R_0^1$ repeatedly, whereas the eigenvectors for the continuous eigenvalues have the wavelike form spreading over $R_j=R_j^0\cup R_j^1$ $(j=0,1,2,\ldots)$ and may be contributed by the invariant sets, where the orbit elements travels over R_j $(j=0,1,2,\ldots)$ unboundedly. Thus the transition at q_c may be interpreted as the transition between the two modes of motions of localized and unbounded.

In the case of the normal regime, i.e., $a_c\leq a<1$, $\phi(q)$ and thus $u(q)$ and $\chi(q)$ are analytic for all q, as shown in Fig. 3. As shown in the Appendix, in this case, only the discrete eigenvalue contributes and no transition takes place.

IV. SUMMARY AND CONCLUDING REMARKS

A simple model of on-off intermittency, the modified random walk model, is introduced to study large deviation properties of on-off intermittency. It is shown that a second-order q-phase transition in $u(q)$ appears for an interval of control parameter with the bifurcation point of on-off intermittency as its end point. This point contrasts with the first-order q-phase transitions at the bifurcation points of chaos [12] and the onset point of type I PM intermittency [16,17], where q-phase transitions are seen only at just before or after the bifurcation points.

The second-order q-phase transition is considered to be a characteristic of on-off intermittency. Indeed, similar calculation performed on the solvable model of on-off intermittency proposed by Hata and Miyazaki [6] leads to a similar second-order q-phase transition. As shown by Fujisaka and Yamada [13], the second-order q-phase transition is also seen in the multiplicative stochastic model of on-off intermittency.

On-off intermittency shows another kind of q-phase tran-

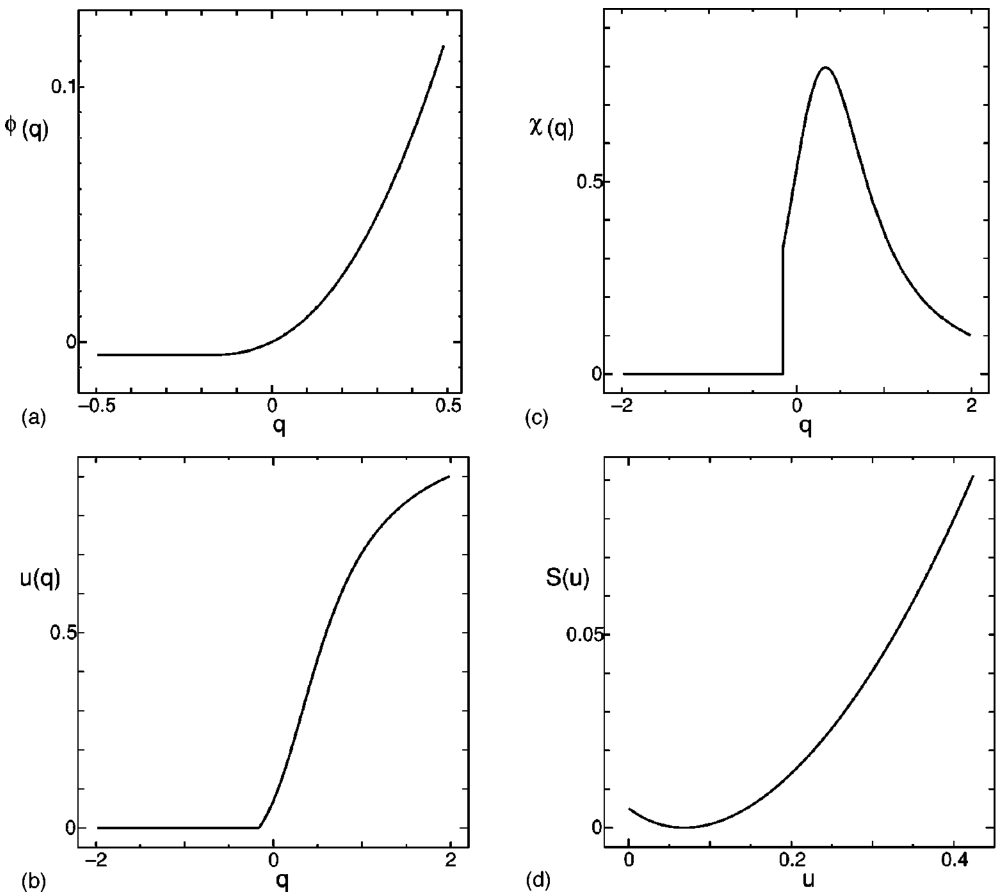

FIG. 2. Thermodynamic structure functions of the modified random walk model in the critical regime with $a=0.6$ and $b=1/\sqrt{3}$. $\chi(q)$ shows a discontinuity at $q=q_c=-0.160\,712$, exhibiting a second-order phase transition at $q=q_c$.

sition as seen in the solvable map of on-off intermittency [6]. The natural invariant density given by Eq. (18) with Eq. (17) leads to the singularity spectrum [18,10]

$$f(\alpha)=\begin{cases} 2+(\alpha-2)/(2-\alpha_0) & \text{if } \alpha_0\leqslant\alpha\leqslant 2,\\ -\infty & \text{otherwise,}\end{cases} \tag{27}$$

where $\alpha_0=1+\min\{1,\log(a'/a)/\log b\}$, exhibiting a linear slope [6], i.e., q-phase transition, for $a_0<a<(1+b)^{-1}$, as shown in Fig. 4. Note that, since $(1+b)^{-1}<a_c=(1+b^2)^{-1}$, q-phase transition in $u(q)$ is observed before the appearance of q-phase transition in $f(\alpha)$ if the control parameter is changed toward the bifurcation point of on-off intermittency. The understanding of geometric structures of on-off intermittency may be needed to clarify the relation between these q-phase transitions. Finally let us mention that, as in our model, on-off intermittency has a close relationship with random walks and, as discussed by Radons [19], the problem of thermodynamic formalism of random walks contains the second-order phase transitions.

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APPENDIX: CALCULATION OF $\phi(Q)$

Let $M_q(n)\equiv\langle e^{nqu_n(X)}\rangle$. Then

$$M_q(n)=\int dX\rho(X)\exp\left\{q\sum_{k=0}^{n-1}u(T^k(X))\right\}$$

$$=\int dX[\mathcal{H}e^{qu}]^n\rho(X), \tag{A1}$$

where

$$[\mathcal{H}e^{qu}]G(X)\equiv\int dY\,\delta(X-T(Y))e^{qu(Y)}G(Y). \tag{A2}$$

Since we are considering the map (6) and $u(X)=R_0(X)$, for a piecewise constant function $f(y)=\Sigma_{j=0}^{\infty}c_jR_j(x,y)$,

$$[\mathcal{H}e^{qu}]f(y)=c_0b'e^q\sum_{j=0}^{\infty}R_j(x,y)$$

$$+\sum_{j=1}^{\infty}[abR_{j-1}(x,y)+a'b^{-1}R_{j+1}] \tag{A3}$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(fR_l)[P_q]_{lk}(R_k)^{-1}R_k(x,y), \tag{A4}$$

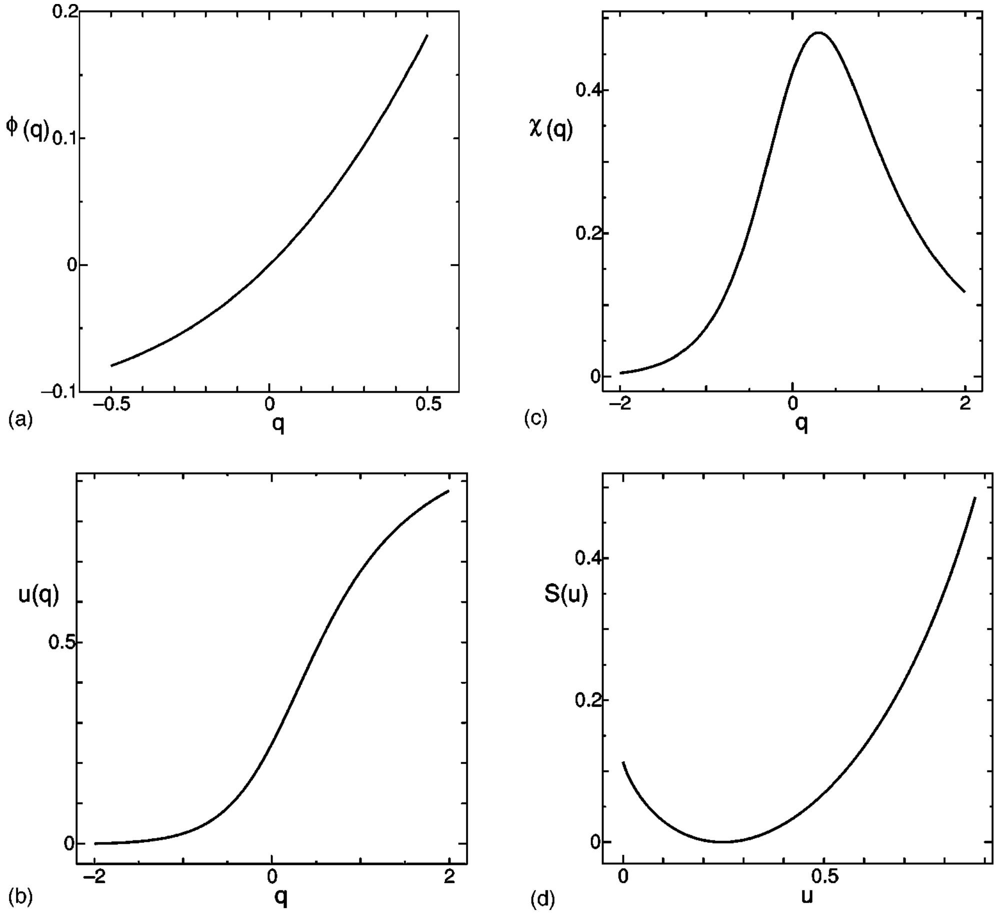

FIG. 3. Thermodynamic structure functions of the modified random walk model in the normal regime with $a=0.95$ and $b=1/\sqrt{3}$.

where P_q has the form

$$P_q=\begin{pmatrix} b' & a & & & & \\ b'b & 0 & a & & 0 & \\ b'b^2 & a' & 0 & a & & \\ b'b^3 & & a' & 0 & a & \\ \vdots & & 0 & \ddots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} e^q & & & \\ & 1 & 0 & \\ & & 1 & \\ & 0 & 1 & \\ & & & \ddots \end{pmatrix} \tag{A5}$$

and P_0 defines an infinite Markov chain. By using Eqs. (18) and (A4), Eq. (A1) leads to

$$M_q(n)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}[P_q{}^n]_{kl}p_l^s. \tag{A6}$$

In order to investigate n dependence of $M_q(n)$, let us consider the eigenvalue problem of P_q. First, we construct a finite Markov chain by approximating P_0 with an $m\times m$ matrix

$$P_0^{(m)}=\begin{pmatrix} b^{(m)} & a & & & & & & 0 \\ b^{(m)}b & 0 & a & & & 0 & & 0 \\ b^{(m)}b^2 & a' & 0 & a & & & & 0 \\ b^{(m)}b^3 & & a' & 0 & a & & & 0 \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ b^{(m)}b^{m-3} & & & & a & 0 & a & 0 \\ b^{(m)}b^{m-2} & & 0 & & & a' & 0 & 1 \\ b^{(m)}b^{m-1} & & & & & & a' & 0 \end{pmatrix}, \tag{A7}$$

where $b^{(m)}=(1-b^m)/b'$, and $P_q^{(m)}$ is naturally introduced by multiplying an $m\times m$ diagonal matrix with diagonal elements $(e^q,1,1,\ldots,1)$ as in Eq. (A5). The left eigenequation of $P_q^{(m)}$ reads

$$e^q b^{(m)}\sum_{i=0}^{m-1} b^i v_{i-1}=\lambda v_0, \tag{A8}$$

$$a v_{i-1}+a' v_{i+1}=\lambda v_i, \quad i=1,\ldots,m-2, \tag{A9}$$

$$v_{m-2}=\lambda v_{m-1}, \tag{A10}$$

where λ denotes the eigenvalue and $(v_0,v_1,\ldots,v_{m-1})$ denotes the eigenvector. Since real and positive eigenvalues are interested, $\lambda>0$ is assumed in the following. For a given λ,

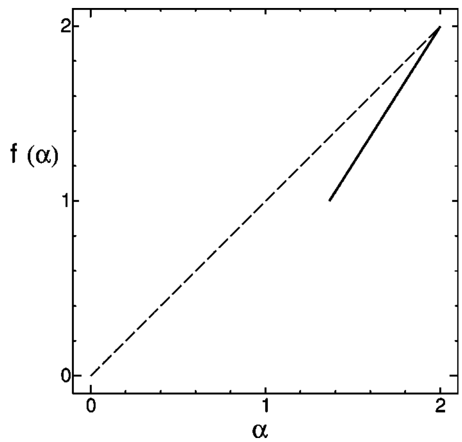

FIG. 4. Singularity spectrum of the invariant measure for the modified random walk model with $a=0.55$ and $b=1/\sqrt{3}$. $f(\alpha)$ has a linear slope for $\alpha_0 \simeq 1.365 \leqslant \alpha \leqslant 2$.

$$v_i = \alpha\mu_+^i + \beta\mu_-^i, \quad i=0,1,\ldots,m-1, \tag{A11}$$

with

$$\mu_\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4aa'}}{2a'} \tag{A12}$$

and constants α and β, satisfies Eq. (A9), thus Eqs. (A8) and (A10) lead to

$$\alpha\left(e^q b^{(m)} \frac{1-b^m\mu_+^m}{1-b\mu_+} - \lambda \right) + \beta\left(e^q b^{(m)} \frac{1-b^m\mu_-^m}{1-b\mu_-} - \lambda \right) = 0, \tag{A13}$$

$$\alpha\mu_+^{m-2}(1-\lambda\mu_+) + \beta\mu_-^{m-2}(1-\lambda\mu_-) = 0, \tag{A14}$$

and furthermore,

$$\beta(\mu_-/\mu_+)^{m-2}(1-\lambda\mu_-)\left(e^q b^{(m)} \frac{1-b^m\mu_+^m}{1-b\mu_+} - \lambda \right) = \beta(1-\lambda\mu_+)\left(e^q b^{(m)} \frac{1-b^m\mu_-^m}{1-b\mu_-} - \lambda \right). \tag{A15}$$

If $\lambda > 2\sqrt{aa'}$, then $\mu_+ > \sqrt{a/a'} > \mu_- > 0$ and, in the limit of $m \to \infty$, Eq. (A15) leads to

$$\frac{e^q b'}{1-b\mu_-} = \lambda, \tag{A16}$$

by abandoning nonphysical solutions $\beta=0$ and $\lambda\mu_+=1$, because the left-hand side of Eq. (A15) goes to 0. Since Eq. (A12) is equivalent to

$$a\mu_-^{-1} + a'\mu_- = \lambda, \quad 0 \leqslant \mu_- \leqslant \sqrt{a/a'}, \tag{A17}$$

the eigenvalue λ is given by the intersecting point of Eqs. (A16) and (A17) on the μ_--λ plane as shown in Fig. 5. If $b < \sqrt{a'/a}$ and $q < q_c \equiv \log\{2(\sqrt{aa'} - ab)/b'\}$, there is no eigenvalue greater than $2\sqrt{aa'}$. Otherwise, by eliminating μ_- in Eqs. (A16) and (A17), the eigenvalue is given as the real root of

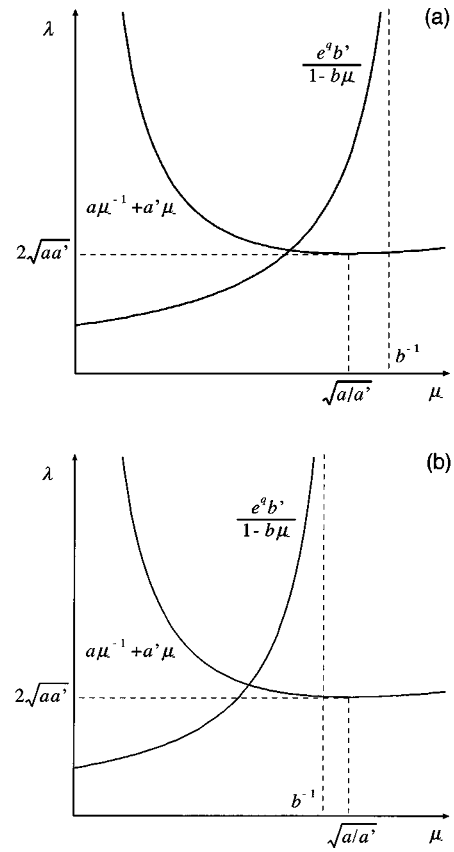

FIG. 5. λ versus μ_- in the critical (a) and normal (b) regime. The crossing point of the two curves determines the eigenvalue. In the critical regime, there is no crossing point satisfying $\mu_- \leqslant \sqrt{a/a'}$ for $q < q_c$; on the other hand, in the normal regime, there is a crossing point for all q.

$$b\lambda^3 - (e^q bb' + ab^2 + a')\lambda^2 + 2e^q a'b'\lambda - (e^q b')^2 a' = 0, \tag{A18}$$

which has a unique real root $\lambda(q)$ greater than $2\sqrt{aa'}$ as it can be seen form Fig. 5.

If $\lambda \leqslant 2\sqrt{aa'}$, by introducing $\theta \in [0,\pi]$, $\mu_\pm$ can be expressed as $\mu_\pm = \sqrt{a/a'}e^{\pm i\theta}$ and $\lambda = 2\sqrt{aa'}\cos\theta$. Without loss of generality, we can set $\alpha = \beta^{-1} = e^{i\psi}$ with real ψ. Eq. (A13) leads to

$$\mathrm{Re}\left\{ e^{i\psi}\left(e^q b^{(m)} \frac{1-b^m(a/a')^{m/2}e^{im\theta}}{1-b\sqrt{a/a'}e^{i\theta}} - 2\sqrt{aa'}\cos\theta \right) \right\} = 0, \tag{A19}$$

which is satisfied with

$$\psi = \frac{\pi}{2} - \arg\left\{ \left(e^q b^{(m)} \frac{1-b^m(a/a')^{m/2}e^{im\theta}}{1-b\sqrt{a/a'}e^{i\theta}} - 2\sqrt{aa'}\cos\theta \right) \right\}. \tag{A20}$$

Equation (A14) leads to

$$a'\cos(\psi + (m-2)\theta) = a\cos(\psi + m\theta), \tag{A21}$$

which has m or $m-1$ solutions with respect to θ. θ's satisfying Eq. (A21) almost uniformly range over $[0,\pi]$; thus, in the limit of $m\to\infty$, a band of continuous eigenvalues over $[-2\sqrt{aa'},2\sqrt{aa'}]$ is formed.

By estimating $M_q(n)$ with the largest eigenvalue $\lambda_{\max}(q)$ of P_q as $M_q(n)\sim(\lambda_{\max}(q))^n$, $\phi(q)$ shown in Sec. III is obtained. In the normal regime $b\geqslant\sqrt{a'/a}$, i.e., $a\geqslant a_c=(1+b^2)^{-1}$, no eigenvalue in the band of continuous eigenvalue does not contribute to $\phi(q)$, whereas, in the critical regime $a_0=1/2<a<a_c$, the largest eigenvalue in the band of continuous eigenvalue determines $\phi(q)$ for $q<q_c$.

Note that for each eigenvector considered above, $|\Sigma_{i=0}^{\infty}v_i p_i^s|<\infty$ holds, because $\alpha\to 0$ as $m\to\infty$ by Eq. (A14) if $\lambda>2\sqrt{aa'}$, and the largest eigenvalue determines $\phi(q)$. It is not the case, but, if $|\Sigma_{i=0}^{\infty}v_i p_i^s|<\infty$ is not guaranteed, there is a possibility that not only the largest eigenvalue determines the growth rate of $M_q(n)$ [19].

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